Introduction to lattice theory
Part I - Suggestions of exercices.

Question 1. For which $n \geq 1$ the lattice of divisor of n is isomorphic to the powerset of $\{0, 1\}$?

Question 2. Give the an example that shows that an order-preserving bijection between lattices might not be an order-isomorphism.

Question 3. Prove that the lexicographic order of two total orders is a total order.

Question 4. Let $\mathbb{N}$ be ordered by divisibility. What is $x \wedge y$? What is $x \vee y$?

Question 5. A *partial map from* X *to* Y is a map $f : S \to Y$ where $S \subseteq X$.

1. Show that the relation $\leq$ defined of the set of partial maps from X to Y by
$$f \leq g \quad \text{if } \mathcal{G}(f) \subseteq \mathcal{G}(g),$$
where $\mathcal{G}(h) := \{(x, h(x)) \mid x \in \text{dom}(h)\}$ for every map h, is a partial order.
2. Give a necessary and sufficient condition for a pair $\{f, g\}$ to have a lest upper bound.
3. Prove that every chain has a least upper bound.

Question 6. Give an example of a poset that contains (at least) three elements x, y, z such that

1. $\{x, y, z\}$ is an antichain,
2. $x \vee y$, $x \vee z$ and $y \vee z$ fail to exist,
3. $\vee\{x, y, z\}$

Question 7. Prove that every order embedding is one-to-one.

Question 8. Let $(X, \leq)$ and $(Y, \leq)$ two posets and $f : X \to Y$. Prove that the following conditions are equivalents.

(i) f is an order-embedding.

(ii) The map $f' : X \to f(X)$ defined as $f'(x) = f(x)$ is an order-isomorphism.

Question 9. Let $(X, \leq)$ and $(Y, \leq)$ two posets and $f : X \to Y$. Prove that the following conditions are equivalents

(i) f is an order-isomorphism.

(ii) f is an order-preserving bijection and f^{-1} is also order-preserving.

Question 10. The aim of this exercise is to show that the lexicographic order of two lattices might not be a lattice. Let X be $\mathcal{P}(\{0,1\}) \times \mathbb{N}$ with lexicographic order. Find two elements of X with no greatest lower bound.

Question 11. Prove that if $(L, \leq)$ is a lattice, then it satisfies the symmetry, associativity, idempotence and asborbtion laws.