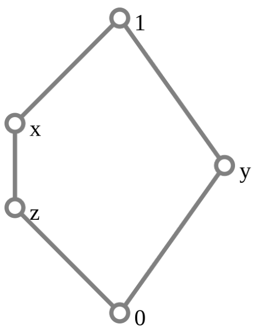
Introduction to lattice theory
Part II - Suggestions of exercices.

Question 1. Let N_5 be the lattice depicted below.



1. Does N_5 satisfies the distributivity equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$?
2. List all the sublattices of N_5.
3. * Let $n \geq 0$. Prove that there is no onto homomorphism from the $\mathcal{P}(\{1, \ldots, n\})$ to N_5.

Question 2. Let $(L, \wedge, \vee)$ and $(L', \wedge, \vee)$ be two lattices and $f \colon L \to L'$. Prove that the following conditions are equivalent.
(i) f is a lattice isomorphism.
(ii) f is an order-isomorphism.

Question 3. Let $(L, \wedge, \vee)$ and $(L', \wedge, \vee)$ be two lattices and $f \colon L \to L'$.

1. Prove that if f is a lattice-homomorphism then it is order-preserving
2. Prove that if f is order-preserving, it needs not be a lattice-homomorphism.

Question 4. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ a family of subsets of X such that $X \in \mathcal{F}$ and $\bigcap \mathcal{F}' \in \mathcal{F}$ for every nonempty $\mathcal{F}' \subseteq \mathcal{F}$. Prove that $\mathcal{F}$ is a complete lattice in which $\bigwedge$ coincides with $\bigcap$ and for every $\mathcal{F}' \subseteq \mathcal{F}$, we have

$$\bigvee \mathcal{F}' = \bigcap \{B \in \mathcal{F} \mid \bigcup \mathcal{F}' \subseteq B\}.$$

Question 5. Deduce from the previous exercise that for any lattice L, the poset of its sublattices with the empty set adjoined is a complete lattice.

Question 6. Let $(X, \leq)$ be a poset. Show that the maps $\alpha, \gamma \colon 2^X \to 2^X$ defined as

$$\alpha(A) = \{y \in X \mid \forall a \in A \; a \leq y\}$$
$$\gamma(A) = \{y \in X \mid \forall a \in A \; y \leq a\}$$

are such that $(2^X, \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (2^X, \subseteq^\partial)$ is a Galois connection.

Question 7. Let C be a closure operator on a poset $(X, \leq)$. Show that an element $x \in X$ is closed if and only if $C(x) = x$.

Question 8. Let C be a closure operator on a poset $(X, \leq)$ and Γ the set of closed elements of C. Define $\alpha \colon X \to \Gamma$ and $\gamma \colon \Gamma \to X$ by

$$\alpha(x) = C(x)$$
$$\gamma(y) = y.$$

Show that $(X, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\Gamma, \leq)$ is a Galois connection.

Question 9. Assume that $(X, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (Y, \leq)$ is a Galois connection. Show that if $S \subseteq P$ is such that $\bigvee S$ exists then $\bigvee \alpha(S)$ exists and $\alpha(\bigvee S) = \bigvee \alpha(S)$.