Introduction to Lattice Theory

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DMATH

Goal

Elementary order and lattice theory for Computer Science.

Order

Poset

X, Y, Z... will be sets

Orders are meant to capture the "less or equal" relations.

Definition. A binary relation $\leq$ on X is an *order* if for all x, y, z

- $x \leq x$ (reflexivity)
- $(x \leq y$ and $y \leq x)$ implies $x = y$ (antisymmetry)
- $(x \leq y$ and $y \leq z)$ implies $x \leq z$ (transitivity)

We call $(X, \leq)$ a *partially ordered set* or *poset*.

Examples

We supercharge $\leq$.

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Example. $(\mathbb{R}, \leq)$ is a poset

How to define $\leq$ on $\mathbb{N}$ and $\mathbb{R}$ is an interesting question.

These are examples of *total orders*, that is, that satisfies

$$\forall x, y \ (x \leq y \text{ or } y \leq x)$$

Example. $(\mathbb{N}, \leq)$ where $a \leq b$ if a divides b is a poset

Examples: natural orders

Example. $(2^X, \leq)$ where 2^X is the powerset of X and $\leq$ is the inclusion relation is a poset.

Definition. A *predicate* on X is a mapping $P\colon X \to \{\mathrm{True}, \mathrm{False}\}$.

Example. $x \geq 2$ is a predicate sur $\mathbb{N}$

Example. Let $\mathbb{P}(X)$ the set of predicates on X. $(\mathbb{P}, \implies)$ is a poset, where

$$P \implies Q \qquad \text{if} \qquad \{x \in X \mid P(X)\} \subseteq \{x \in X \mid Q(X)\}$$

Hasse diagrams

A (finite) poset $(X, \leq)$ can be represented by a diagram depicted according to the following rules:

1. Elements $x \in X$ are represented by dots.
2. If $x \leq y$ then x is represented below y and both are connected by a straight line unless ...

2 bis. the relation $x \leq y$ can de deduced from $x \leq z$ and $z \leq y$ for some z.

It is called the *Hasse diagram* of $(X, \leq)$

Hasse diagrams: examples

Draw the Hasse diagrams of $(\mathbb{N}, \leq)$, the divisors of 12 ordered by divisibility and the powerset $2^{\{0,1\}}$.

Order dual and strict order

Definition. The *(order) dual* of $(X, \leq)$ is the poset $(X, \leq^\partial)$ defined as

$$x \leq^\partial y \qquad \text{if} \qquad y \leq x.$$

How to obtain the Hasse diagram of $(X, \leq^\partial)$ from that of $(X, \leq)$?

Definition. We write $x < y$ if $(x \leq y$ and $y \not\leq x)$.

Such a $<$ is called a *strict partial order*.

Mappings

There are different natural classes of mappings betwen posets.

Definition. A map $f\colon (X, \leq) \to (Y, \leq)$ is

- *order-preserving* if $x \leq y \implies f(x) \leq f(y)$ for all x, y

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- an *order-embedding* if $x \leq y \iff f(x) \leq f(y)$ for all x, y
- an *order-isomorphism* if f is an onto order-embedding.

Isomorphic posets can't be distinguished from the perspective of order theory.

Mappings: example

Example. Give two integers whose divisors posets are isomorphic

Example. The map $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 2x$ is an embedding.

Example. The powerset $(2^X, \subseteq)$ and the predicate poset $(\mathbb{B}, \implies)$ are isomorphic.

Poset consctructions: disjoint union

Definition. The *disjoint union* $(X, \leq) \mathbin{\dot\cup} (Y, \leq)$ is the poset defined on $X \mathbin{\dot\cup} Y$ by

$$s \leq t \qquad \text{if } (s, t \in X \text{ and } s \leq t) \text{ or } (s, t \in Y \text{ and } s \leq t)$$

Example.

Poset constructions: linear sum

Definition The *linear sum* $(X, \leq) \oplus (Y, \leq)$ is the poset defined on $X \uplus Y$ by

$$s \leq t \qquad \text{if} \qquad (s, t \in X \text{ and } s \leq t) \\ \text{or } (s, t \in Y \text{ and } s \leq t) \\ \text{or } (s \in X \text{ and } t \in Y).$$

Example.

Poset consctructions: Cartesian product

Given $(X, \leq)$ and $(Y, \leq)$ how to define a poset on $X \times Y$?

Definition. The *pointwise order* on $X \times Y$ is defined by

$$(x, y) \leq (x', y') \qquad \text{if} \qquad (x \leq x' \text{ and } y \leq y')$$

The *lexicographic order* on $X \times Y$ is defined by

$$(x, y) \leq (x', y') \qquad \text{if} \quad \big(x < x' \text{ or } (x = x' \text{ and } y \leq y')\big).$$

Exercise. Check that the lexicographic and pointiwse orders are orders.

Examples.

Exercise. Prove that the lexicographic order of total orders is a total order.

Bounds

Upper and lower bounds

Definition. Let $(X, \leq)$ be a poset and $S \cup \{u, \ell\} \subseteq X$.

1. u is an *upper bound* of S if $s \leq u$ for all $s \in S$
2. ℓ is a *lower bound* of S if $\ell \leq s$ for all $s \in S$

Examples.

Best bounds

One Bound to rule them all, One Bound to find them.

Definition. Let $(X, \leq)$ be a poset and $S \cup \{u, \ell\} \subseteq X$.

1. u is a *least upper bound (lub)* of S if u is an upper bound of S and $u \leq u'$ for every upper bound u' of S.
2. ℓ is a *greatest lower bound (glb)* of S if ℓ is a lower bound and $\ell' \leq \ell$ for every lower bound ℓ' of S.

Lub and glb are the best bounds.

Definition. If $S \subseteq X$ has a lup (glb, resp.) α and $\alpha \in S$, we say that S has a *greatest element* (*smallest element*, resp.) α.

Examples

Best bounds are unique

Lemma. If u and u' are two lub of S in $(X, \leq)$ then $u = u'$.

Proof. We have $u \leq u'$ and $u' \leq u$.

Unicity also holds for glb.

Bottom and top

Definition. A *top element* $\top$ *of* $(X, \leq)$ is a lup of $(X, \leq)$.

A *bottom element* $\perp$ *of* $(X, \leq)$ is a glb of $(X, \leq)$.

Sometimes top and bottom elements are denoted by 1 and 0, respectively.

Examples.

Powersets have top and bottom elements.

$(\mathbb{Z}, \leq)$ has neither a greatest element, nor a least element.

Lattices

Definition. A *lattice* is a poset in which every pair $\{x, y\}$ has a lub and a glb.

We denote by $x \wedge y$ and $x \vee y$ the glb and lub of $\{x, y\}$.

A lattice is *bounded* if it has a bottom and a top element.

Examples

Lattices as algebraic structures

A lattice $(L, \leq)$ can be seen as an algebraic structure $(L, \wedge, \vee)$ equipped with two binary operations

$$\wedge, \vee, : L \times L \to L,$$

It satisfies the following equations:

$$x \wedge y = y \wedge x \qquad \text{(symmetry)} \tag{1}$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad \text{(associativity)} \tag{2}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{(distributivity)} \tag{3}$$

$$x \wedge x = x \qquad \text{(idempotence)} \tag{4}$$

$$x \vee (x \wedge y) = x \qquad \text{(absorption)} \tag{5}$$

and their dual.

Lattices as algebraic structures

A bounded lattice $(L, \leq)$ can be seen as an algebraic structure $(L, \wedge, \vee 0, 1)$ equipped with two binary operations

$$\wedge, \vee, : L \times L \to L,$$

and constants 0 and 1.

It satisfies the following equations:

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$$x \vee (x \wedge y) = x \qquad \text{(absorption)} \tag{5}$$

$$x \wedge 1 = 1, \quad x \vee 0 = 0$$

and their dual.

Lattices as posets $\equiv$ lattices as algebra

Proposition. In a lattice $(L, \leq)$, we can recover $\leq$ from $\wedge$ or $\vee$:

$$a \leq b \quad \Longleftrightarrow \quad a \wedge b = a \quad \Longleftrightarrow \quad a \vee b = b$$

Proposition. If $(L, \wedge, \vee)$ satisfies equations (1) - (5) then the relation $\leq$ defined as

$$a \leq b \quad \text{if } a \wedge b = a$$

is a lattice order. The glb and lub operations in $(L, \leq)$ coincide with $\wedge$ and $\vee$, respectively.

We use the order-theoretic and algebraic perspectives interchangeably.

Lattices and poset constructions

The disjoint union of lattices

Lattices and poset constructions

The disjoint union of lattices is not a lattice

The linear sum of lattices

Lattices and poset constructions

The disjoint union of lattices is not a lattice

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The lexicographic order on product of lattices

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The lexicographic order on product of lattices might not be a lattice (see exercises)

The pointwise order on product of lattices

Lattices and poset constructions

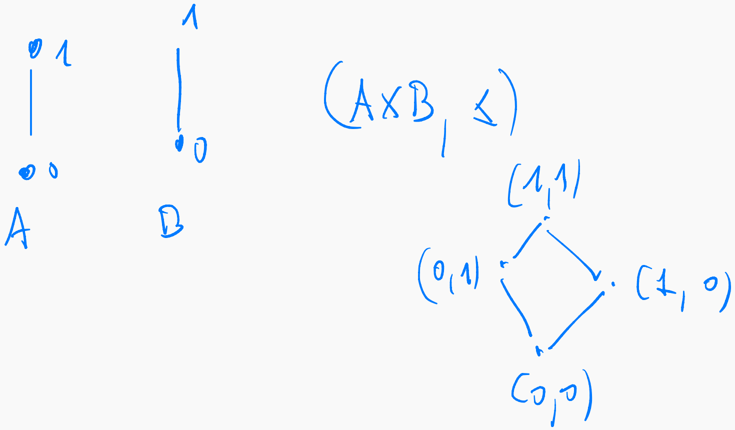
The disjoint union of lattices is not a lattice

The linear sum of lattices is a lattice

The lexicographic order on product of lattices might not be a lattice (see exercises)

The pointwise order on product of lattices *is* a lattice. Operations $\vee$ and $\wedge$ are computed pointwise.

Examples



Lattice constructions: sublattices

Definition. A subset S of a lattice $(L, \wedge, \vee)$ is called a *sublattice* if $x \wedge y$ and $x \vee y$ belongs to S for every $x, y \in S$.

Sublattices are lattices. They inherit their order and operations from their parent lattice.

Sublattice: Examples

Suitable maps between lattices

Definition. A map $f\colon (L, \wedge, \vee) \to (L', \wedge, \vee)$ is a *lattice homomorphism* if for every $a, b \in L$

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

A bijective lattice homomorphism is called a *lattice isomorphism*.

Examples.

Complete lattices: the best case scenario

Definition. A lattice $(L, \leq)$ is *complete* if every subset S of L has a lup and a glb, denoted by

$$\bigvee S \qquad \text{and} \qquad \bigwedge S,$$

and called the *supremum* and *infimum* of S, respectively.

A complete lattice $(L, \leq)$ has all joins and all meets. In particular,

$$\bigwedge L = \bigvee \varnothing \quad \text{and} \quad \bigvee L = \bigwedge \varnothing,$$

which are bottom and top element of $(L, \leq)$

Examples

Powerset lattices are complete.

$$\bigwedge S = \bigcap S \qquad \bigvee S = \bigcup S$$

Let $(L, \leq)$ be a complete lattice

X any set

$X \to L \equiv$ mappings from X to L

Define $\leq$ on $X \to L$ by

$$f \leq g \quad \text{iff} \quad \forall x \; f(x) \leq g(x)$$

Then $(X \to L, \leq)$ is complete

Completeness can be summarized

Lemma. A lattice $(L, \leq)$ is complete if and only if it has a top element $\top$ and every nonempty subset S of L a glb.

Proof. $\Leftarrow$ We have $\bigvee \varnothing = \top$ and if $T \subseteq L$ is nonempty

$$\bigvee T = \bigwedge \{\ell \in L \mid \forall t \in T \;\; t \leq \ell\}$$

Closure operator

Definition. A map $C\colon (X, \leq) \to (X, \leq)$ is a *closure operator* if for all $x \in X$

1. $x \leq C(x)$
2. $x \leq y \implies C(x) \leq C(y)$
3. $C(C(x)) = C(x)$.

Elements of the form $C(x)$ for some x are call *closed*.

Closure operators: Examples

Definition A subset S of $(X, \leq)$ is a *downset* if

$$(s \in S \quad \text{and} \quad x \leq s) \qquad \Longrightarrow \qquad x \in S.$$

For every susbet Y of X we set

$$Y{\downarrow} := \{x \in X \mid \exists y \in Y \; x \leq y\}.$$

Example. The map $-{\downarrow}$ is a closure operator on the powerset of X.

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Example. The linear span is a closure operator on the powerset of a vector space.

From Closure Opeators to Complete lattices

Theorem. For any complete lattice $(L, \leq)$ define $C\colon 2^L \to 2^L$ by

$$C(A) := \bigcap\{x\downarrow \mid A \subseteq x\downarrow\}.$$

Then,

1. C is a closure operator.

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Then,

1. C is a closure operator.
2. The set Γ of closed elements of C is a complete lattice for $\subseteq$ and

$$\bigwedge S = \bigcap S \qquad \text{and} \qquad \bigvee S = C(\bigcup S)$$

   for every $S \subseteq \Gamma$.

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Then,

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2. The set Γ of closed elements of C is a complete lattice for $\subseteq$ and
$$\bigwedge S = \bigcap S \qquad \text{and} \qquad \bigvee S = C(\bigcup S)$$
for every $S \subseteq \Gamma$.
3. The lattices $(\Gamma, \leq)$ and $(L, \leq)$ are isomorphic.

Fixpoints

Fixpoints

In general, a fix point of $F\colon X \to X$ is a $x \in X$ such that

$$F(x) = x.$$

Fixpoints are commonly used in CS in the context of posets.

We give a few existence and computation results for fixpoints.

Fixpoints in posets

$F\colon (X,\leq) \to (X,\leq)$

Notation.

$$\mathrm{fix}(F) := \{x \in X \mid F(x) = x\}$$
$$\mathrm{post}(F) := \{x \in X \mid x \leq F(x)\}$$
$$\mathrm{pre}(F) := \{x \in X \mid F(x) \leq x\}$$

These sets might be empty or nonempty, bounded or unbounded...

If $\bigvee \mathrm{fix}(F)$ exists and belongs to $\mathrm{fix}(F)$, it is denoted by $\nu(F)$.

If $\bigwedge \mathrm{fix}(F)$ exists and belongs to $\mathrm{fix}(F)$, it is denoted by $\mu(F)$.

Fixpoints in Complete Lattices

Theorem (Knaster - Tarski) Let $(L, \leq)$ be a complete lattice and $F\colon (L, \leq) \to (L, \leq)$ be an order-preserving map.

1. F has a least fixpoint $\mu(F)$ and a greatest fixpoint $\nu(F)$.
2. We have

$$\mu(F) = \bigwedge \mathrm{pre}(F),$$
$$\nu(F) = \bigvee \mathrm{post}(F).$$

Proof.[...]

This theorem does no help to compute or approximate fixpoints.

Complete Partial Orders (CPOs)

CPOs are the general environnement to state fixpoint theorems.

Definition. A subset D of $(X, \leq)$ is *directed* if every pair of elements of D has an upper bound in D.

Chains are examples of directed set.

Definition. A poset $(X, \leq)$ is a *complete partial order (CPO)* if

1. it has a bottom element $\bot$,
2. $\bigvee D$ exists for every directed $D \subseteq X$.

Complete lattices are instances of CPO.

Recognizing CPOs

Theorem ($).

$(P, \leq)$ is a CPO if and only if each chain has a lub (in P).

Continuous transformations

Definition. A map $f\colon (X,\leq) \to (Y,\leq)$ between CPOs is *continuous* if for every directed $D \subseteq X$,

$$f(D) \text{ is directed} \qquad \text{and} \qquad f(\bigvee D) = \bigvee f(D).$$

Continuous implies order-preserving but the converse is false.

Definition. $(X,\leq)$ satisfies the *ascending chain condition (AAC)* if every increasing sequence of P is eventually constant.

Lemma If $(X,\leq) \models$ AAC, then any order-preserving $f\colon (X,\leq) \to (Y,\leq)$ is continuous.

Next Fixpoint Theorem

Theorem. Let $F\colon (X, \leq) \to (X, \leq)$ be an order-preserving map on a CPO and set

$$\alpha := \bigvee \{ F_n(\perp) \mid n \geq 0 \}$$

1. If α is a fixpoint then it is the least fixpoint.
2. If F is continuous then it is a least fixpoint $\mu(F)$ and $\mu(F) = \alpha$.