Introduction to Constraint Programming

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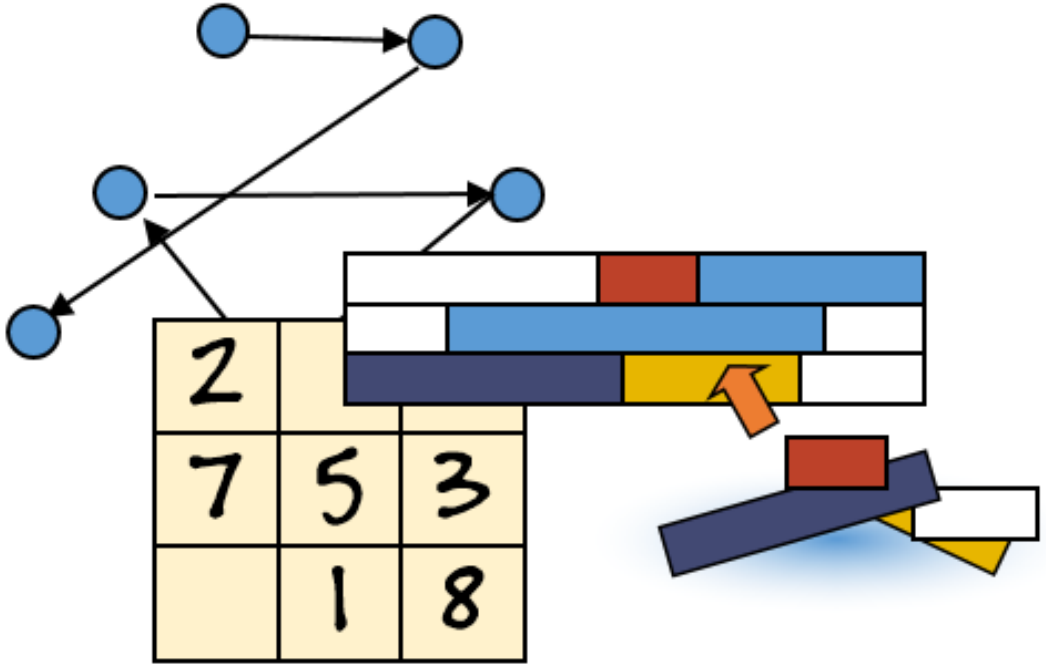
Agenda

1. Constraint satisfaction problem
2. Minizinc
3. Solving algorithm
4. Global constraint
5. Real problem



Constraint Satisfaction Problem (CSP)

$Triplet\ \ < X, D, C >$

$X: Set\ of\ variables$

$D: Domains\ of\ variables$

$C: Set\ of\ constraints$

$Example$

$< \{x, y\}, \left\{\{0,1,2\}_x, \{2,3,4\}_y\right\}, \{x \neq y, x > 1\} >$

$Variables: \{x, y\}$

$Domain:\ x: \{0,1,2\}, y: \{2,3,4\}$

$Constraints: x \neq y,\ \ x > 1$

Constraint Satisfaction Problem (CSP)

Assignment is a function $asn: X \rightarrow \mathbb{Z}$

Example:

- $asn: \{x \rightarrow 0, y \rightarrow 0\}$
- $asn: \{x \rightarrow 2, y \rightarrow 4\}$

A **solution** is an assigment that satisfies all the constraints

$asn: \{x \rightarrow 2, y \rightarrow 4\}$ satisfies $x \neq y,\ x > 1$

A problem can have several solutions, when you want to find the best solution based in one parameter or objective we said it is an optimization problem.

$Example$

$Variables: \{x, y\}$

$Domain:\ x: \{0,1,2\}, y: \{2,3,4\}$

$Constraints: x \neq y,\ x > 1$

MiniZinc: Basic structure

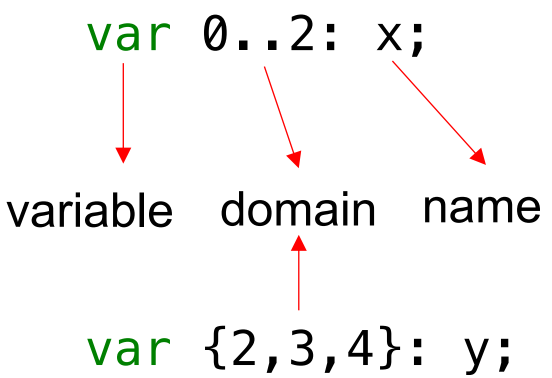
Is modeling language to specify a CSP. https://www.minizinc.org/

$Variables: \{x, y\}$

$Domain:\ x: \{0,1,2\}, y: \{2,3,4\}$

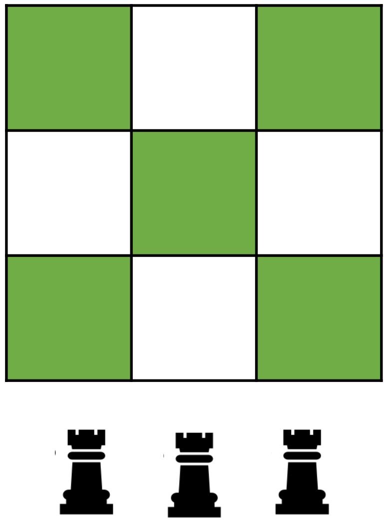
$Constraints: x \neq y,\ \ x > 1$

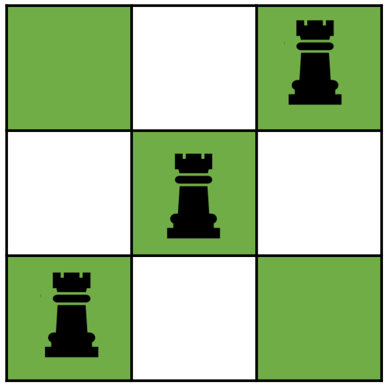
```
var 0..2: x;
var {2,3,4}: y;
constraint x != y;%arithmetic operators, {>,>=,=<,<,!=,=}
constraint x > 1;
solve satisfy;
```



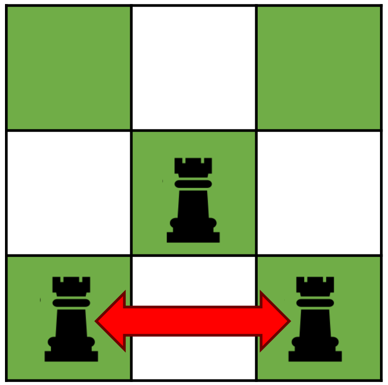
MiniZinc: 3-towers

Can you put 3 towers in a chessboard of 3x3, in a way that they cannot attack each other?





This is a solution



This is **not** a solution

MiniZinc: 3-towers model

Model: Variables, Domains, Constraints

Variables: $\{T_1, T_2, T_3\}$

Domain: $T_1\colon\{0,1,2\}, T_2\colon\{0,1,2\}, T_3\colon\{0,1,2\}$. Domain represents the column

Constraints: $T_1 \neq T_2, T_1 \neq T_3, T_2 \neq T_3$

```
var 0..2: T1;
var 0..2: T2;
var 0..2: T3;
constraint T1 != T2;
constraint T1 != T3;
constraint T2 != T3;
solve satisfy;
```

```
Running 3_towers.mzn
T1 = 2;
T2 = 1;
T3 = 0;
----------
T1 = 1;
T2 = 2;
T3 = 0;
----------
T1 = 2;
T2 = 0;
T3 = 1;
----------
T1 = 0;
T2 = 2;
T3 = 1;
----------
T1 = 1;
T2 = 0;
T3 = 2;
----------
T1 = 0;
T2 = 1;
T3 = 2;
----------
==========
```

MiniZinc: N-queens

Queens -> Row, Column, Diagonal
N -> Parameter not fixed

Exercise:
1. Complete with the missing constraints.
2. Is it possible to get a solution with n=3?
3. How many queens can you solve in less than 5 seconds?

```
int: n=?;
array[1..n] of var 1..n: queens;

constraint forall(i in 1..n, j in i+1..n)
( queens[i]+i != queens[j]+j
/\ queens[i]-i != queens[j]-j);

solve satisfy;
```

→ Generate a conjunction of constraints /\

Solving algorithm

Naive algorithm: Enumerate all possible combination of values

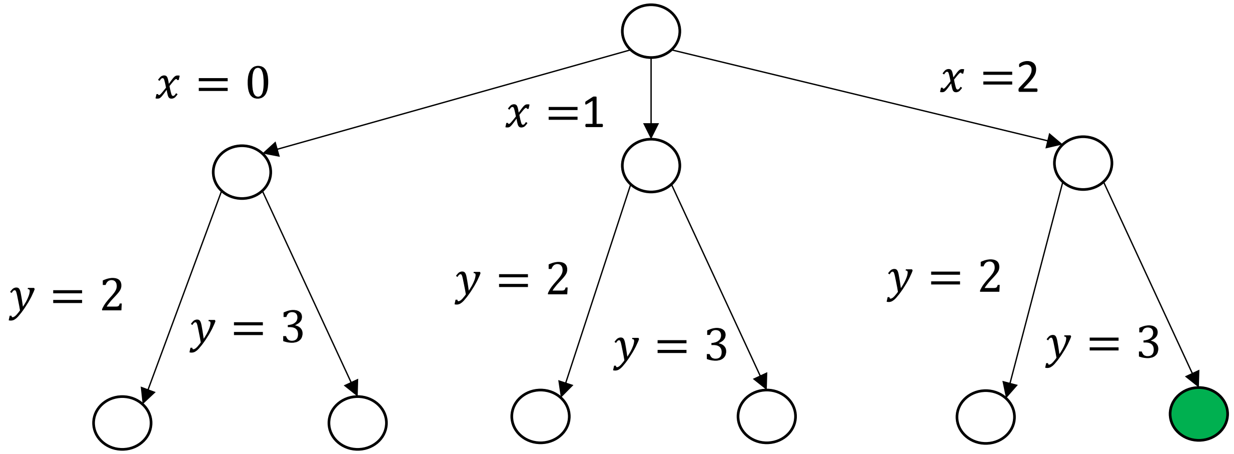
$Variables: \{x, y\}$

$Domain:\ x: \{0,1,2\}, y: \{2,3\}$

$Constraints: x \neq y,\ \ x > 1$

$x = 0, y = 2$
$x = 0, y = 3$
$x = 1, y = 2$
$x = 1, y = 3$
$x = 2, y = 2$
$x = 2, y = 3$

We can get all the possible combinations with the search tree



Solving algorithm

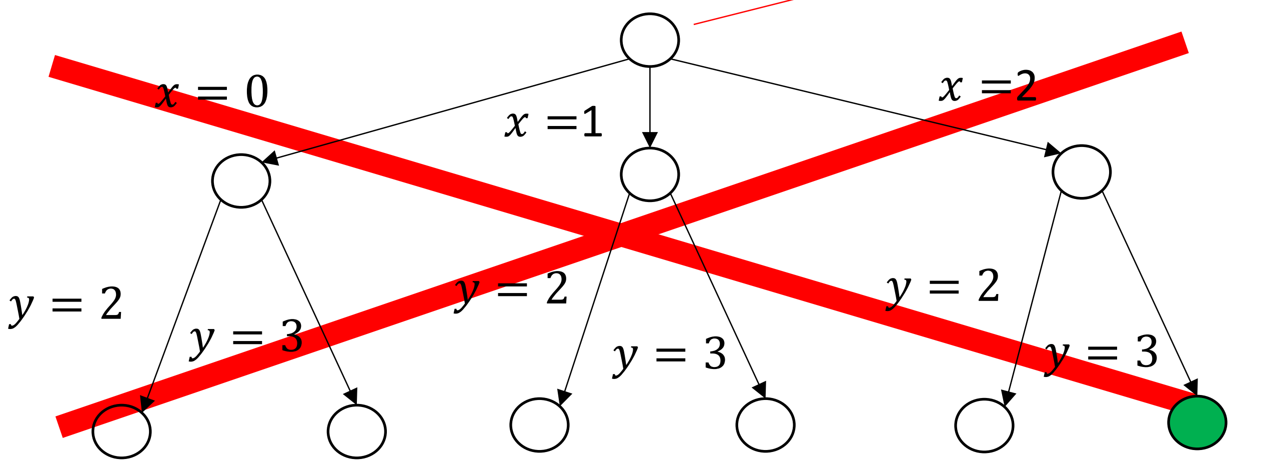
CP solvers perform an inference step, called **propagation**, in each node

- Given the domains and **one** constraint, can we remove values from the domains?

$Variables: \{x, y\}$

$Domain: \ x: \{0,1,2\}, y: \{2,3\}$

$Constraints: x \neq y, \ \ x > 1$



$x \neq y$:
$x: \{0,1,2\}, y: \{2,3\}$

$x > 1$:
$x: \{~~0,1~~, \mathbf{2}\}, y: \{2,3\}$

$x \neq y$:
$x: \{2\}, y: \{2,3\}$

All constraints are satisfied, search is not necessary.

Solutions:
$x = 2, y = 3$

Solving algorithm

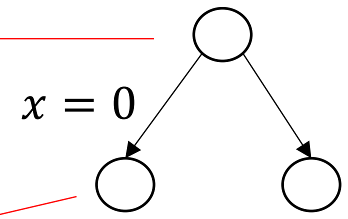
Not always we can find the solutions without searching

$Variables: \{x, y, z\}$

$Domain:\ x: \{0,1\}, y: \{0,1\}, z: \{0,1\}$

$Constraints: x \neq y, y \neq z$

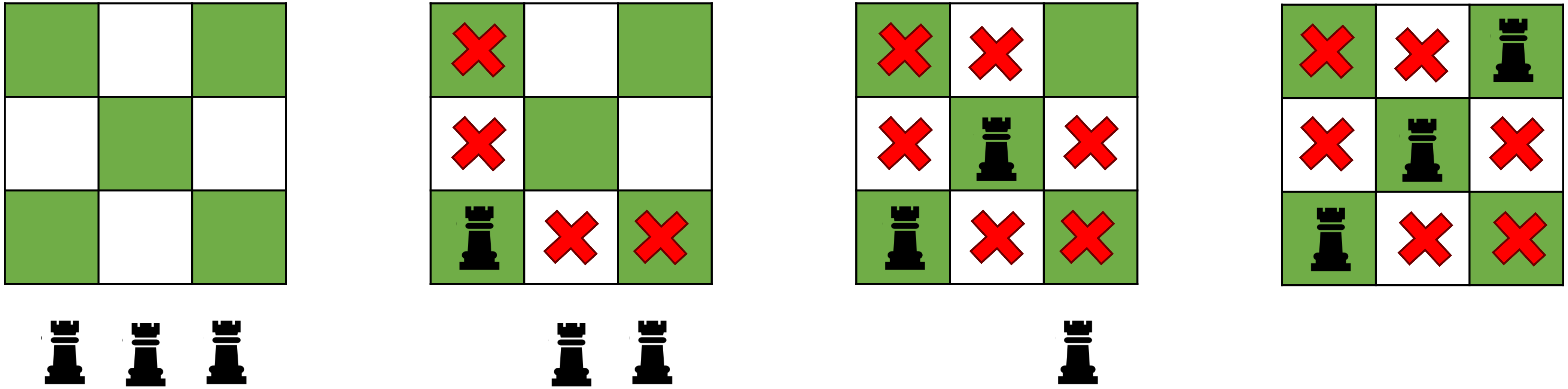
$x \neq y$ and $y \neq z$ do not produce a solution, search



1- $x = 0$ given $x \neq y$ we have $y = 1$, $z: \{0,1\}$
2- $y = 1$ given $y \neq z$ we have $z = 1$
3- Solution $\{x = 0, y = 1, z = 0\}$

MiniZinc: 3-towers

Can you put 3 towers in a chessboard of 3x3, in a way that they cannot attack each other?



Solving algorithm

The interleaving of propagate and search is called ***propagate-and-search*** algorithm.

$solve(< X, D, C >)$

$D' \leftarrow \text{propagate}(< X, D, C >)$

$\textbf{if } \forall d \in D', |d| = 1$

$\textbf{return } \{D'\}$ // we found a solution

$\textbf{if } \exists d \in D', |d| = 0$

$\textbf{return } \{\,\}$ // there are no solution

$\{L, R\} \leftarrow \text{split}(D')$

$\textbf{return } solve(< X, L, C >) \cup solve(< X, R, C >)$ // search

Global constraint

Reasoning locally on constraints is not always the most efficient way to solve the problem

- Global constraints help to reason more globally, find infeasibilities earlier, prune domain better.

$Variables: \{x, y, z\}$

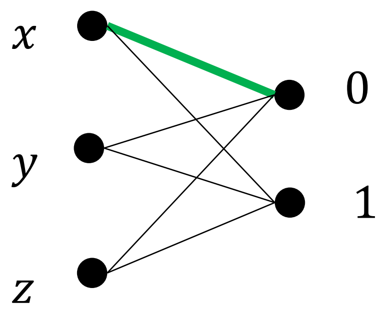
$Domain:\ x: \{0,1\}, y: \{0,1\}, z: \{0,1\}$

$Constraints: x \neq y, y \neq z, y \neq z$

We cannot detect failure when we apply the constraints individually. But with the global constraint $alldifferent$ we can.

Global constraint - alldifferent

$alldifferent(x_1, x_2, \ldots, x_n)$ semantically equivalent to $\{\, x_i \neq x_j \; for\; all\; i \neq j \,\}$ but provides a more efficient propagation algorithm (graph matching).



$Variables: \{x, y, z\}$

$Domain:\; x: \{0,1\}, y: \{0,1\}, z: \{0,1\}$

$Constraints: x \neq y, y \neq z, y \neq z$

Matching: Subset of edges s.t. no common endpoint exists for any pair of edges.

Global constraint - alldifferent

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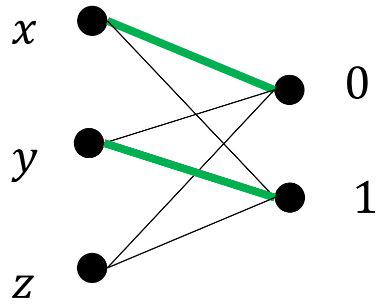


$Variables: \{x, y, z\}$

$Domain: \ x: \{0,1\}, y: \{0,1\}, z: \{0,1\}$

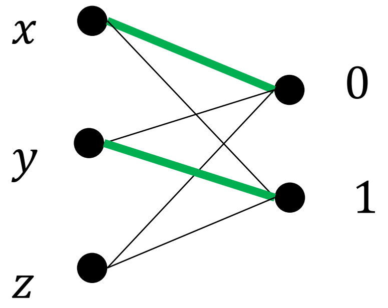
$Constraints: x \neq y, y \neq z, y \neq z$

Matching: Subset of edges s.t. no common endpoint exists for any pair of edges.

Maximum matching: A matching that cannot be augmented by any edge.

Global constraint - alldifferent

$alldifferent(x_1, x_2, \ldots, x_n)$ semantically equivalent to $\{ x_i \neq x_j \; for \; all \; i \neq j \}$ but provides a more efficient propagation algorithm (graph matching).



$Variables: \{x, y, z\}$

$Domain: \; x: \{0,1\}, y: \{0,1\}, z: \{0,1\}$

$Constraints: x \neq y, y \neq z, y \neq z$

Matching: Subset of edges s.t. no common endpoint exists for any pair of edges.

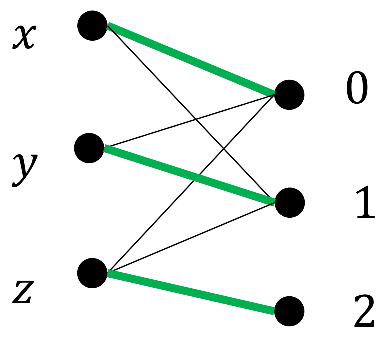
Maximum matching: A matching that cannot be augmented by any edge.

Solution of alldifferent: Maximum matching covering a set of variables.

Infeasible. The cardinality of maximum matching (2) is smaller than the number of variables (3)

Global constraint - alldifferent

Besides detecting infeasibility earlier, can assign values earlier



$z: \{\sout{0,1},2\}$

$Variables: \{x, y, z\}$

$Domain: \; x: \{0,1\}, y: \{0,1\}, z: \{0,1,2\}$

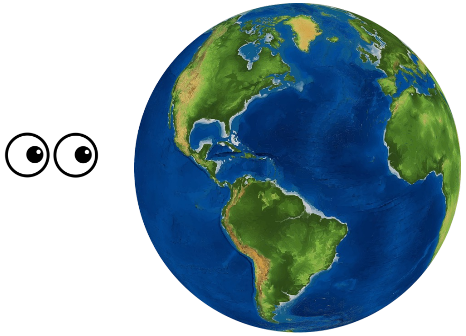
$Constraints: x \neq y, y \neq z, y \neq z$

Global constraint - alldifferent

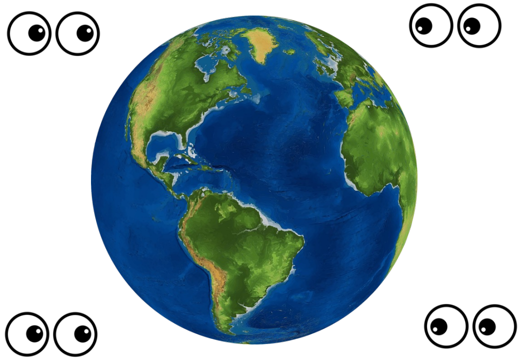
Exercise: Try alldifferent in N-queens and check the efficiency.

```
include "alldifferent.mzn";

int: n=200;
array[1..n] of var 1..n: queens_alldiff;

constraint alldifferent(queens_alldiff);
constraint alldifferent([queens_alldiff[i]+i | i in 1..n]);
constraint alldifferent([queens_alldiff[i]-i | i in 1..n]);

solve satisfy;
```

Satellite image selection problem (SIMS)



2014
192 EO satellites



2021
971 EO satellites

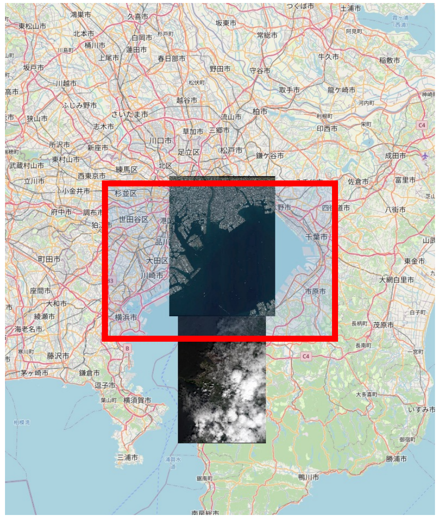
>100 TB of satellite imagery per **day**

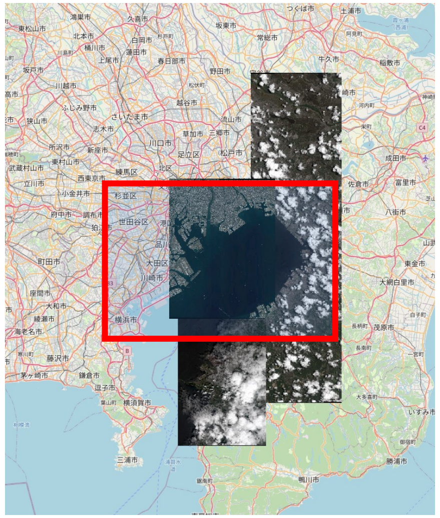
Satellite images: Planet Labs, Inc.

Satellite image selection problem (SIMS)

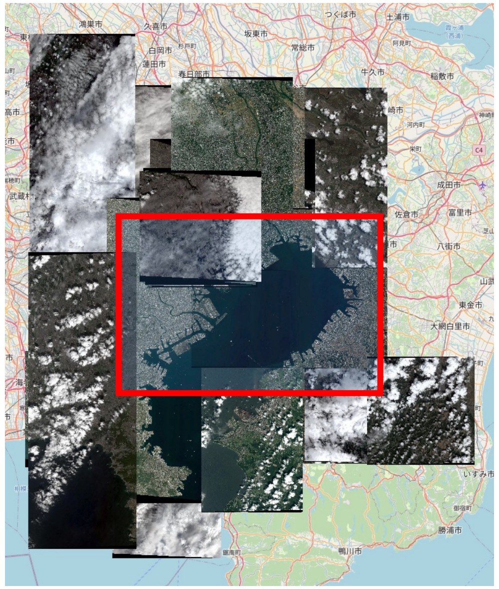
To cover large areas we need several images

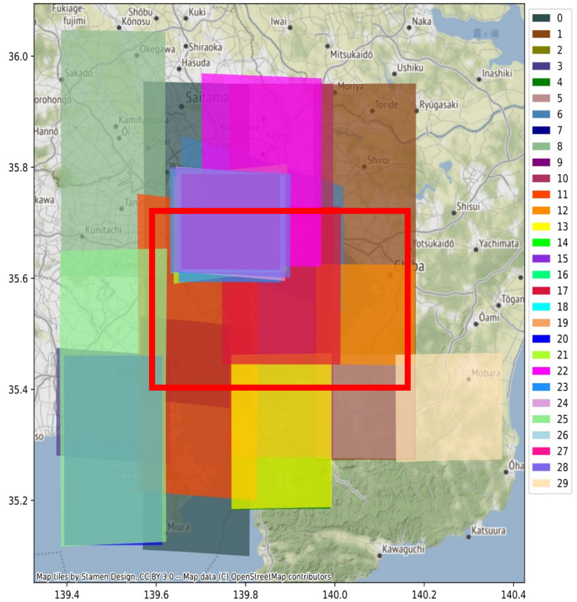
Mosaic



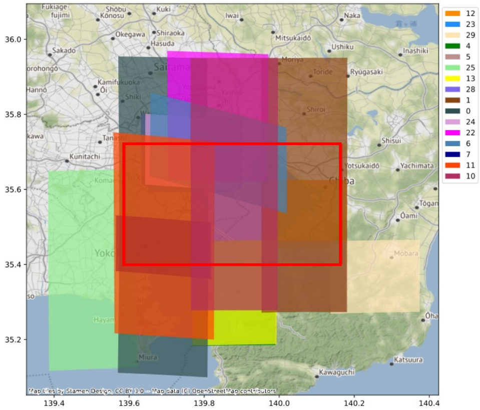


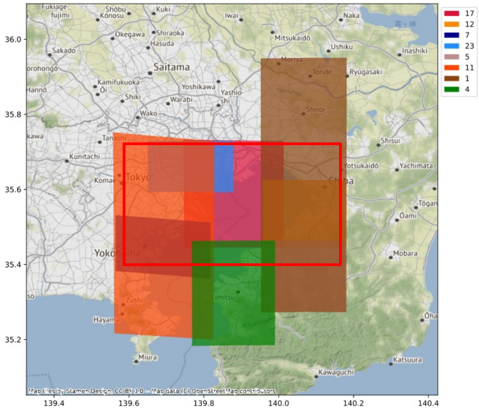
Satellite image selection problem (SIMS)

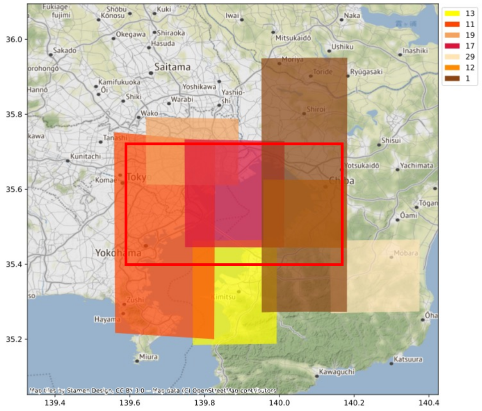


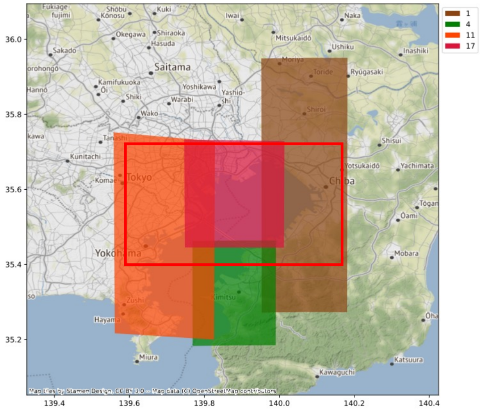


Which combination?
NP-Hard
Enumeration: 2^n

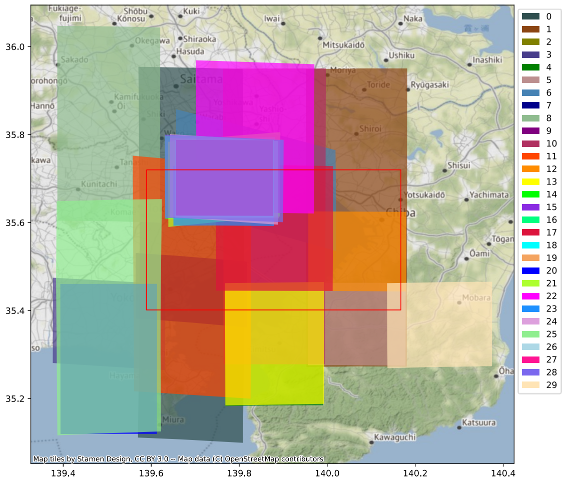






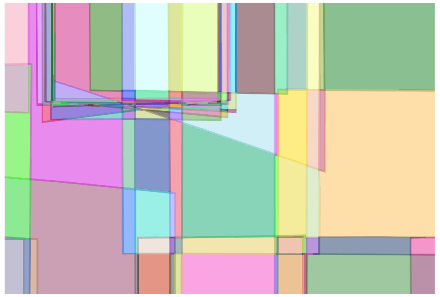


Satellite image selection problem (SIMS)



Remove the area of images outside AOI

Find all intersections

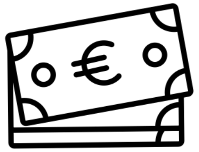


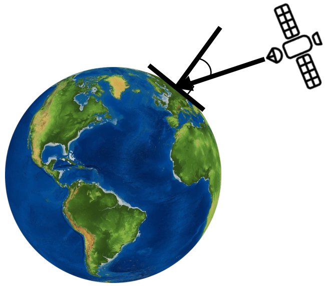
The cover constraint and cost can be modeled as the classical *weighted set cover* problem

Universe = Union of intersections (parts)
Images -> Sets with parts and weight = cost

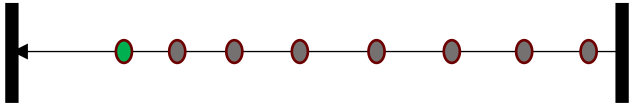
Satellite image selection problem (SIMS) - Model

Multi-objective problem:
- Cost
- Clouds
- Resolution
- Incidence angle



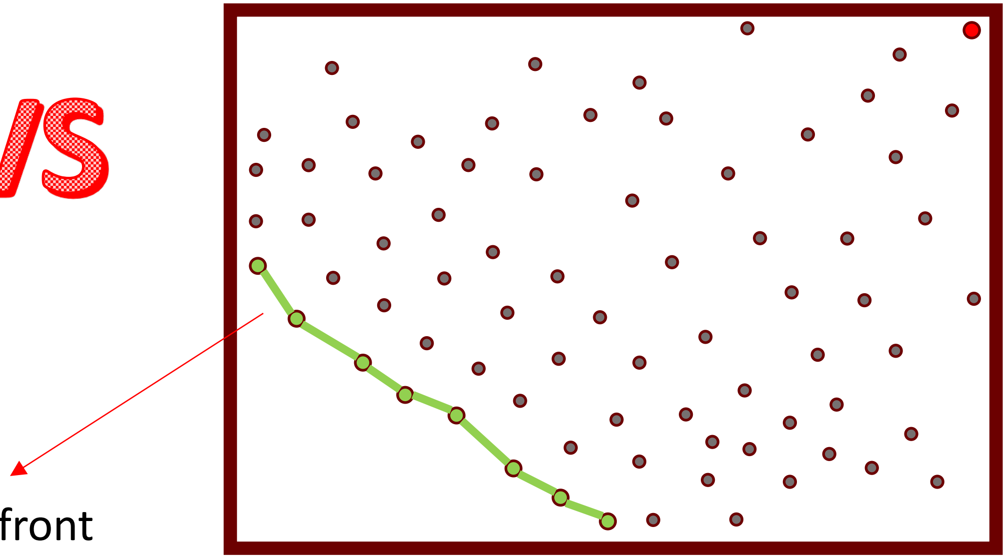


Single objective



VS

Multiobjective



Satellite image selection problem (SIMS) - Model

$Variables: \{taken_i | i = 1, .. n\}$

$Domain: taken_i: \{false, true\}$

$Constraints: cover$

$Objectives: \; cost, resolution, incidence$

Cover constraint:

$$\bigvee_{i:u \in Img_i} taken_i = \text{true}, \qquad \text{for all } u \in Universe$$

```
constraint forall(u in UNIVERSE)(exists(i in IMAGES)(taken[i] /\ u in images[i]));
```

Satellite image selection problem (SIMS) - Model

Cost:

$$min \sum_{i \in Img} cost_i * taken_i$$

```
var int: total_cost = sum(i in IMAGES)(costs[i] * taken[i]);
```

Satellite image selection problem (SIMS) - Model

Resolution:

$$min \sum_{u \in Universe} min\,\{ R_i \mid u \in P_i, taken_i = \text{true}\}$$

```
var int: max_resolution = sum(u in UNIVERSE)(min(i in IMAGES where u in images[i] /\ taken[i])(resolution[i]));
```

Incidence angle:

$$min\{ max\{ taken_i * Inc_i \mid i \ \in Img\}\}$$

```
var int: max_incidence = max(i in IMAGES)(taken[i] * incidence_angle[i]);
```

SnT

Parallel Computing and Optimisation Group

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