Octagon Abstract Domain

Session 6

Thibault Falque

Abstract Interpretation Workshop – 20th June 2024

University of Luxembourg

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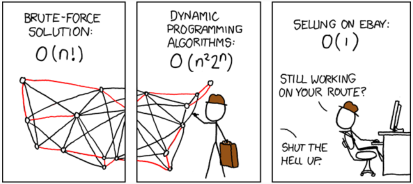
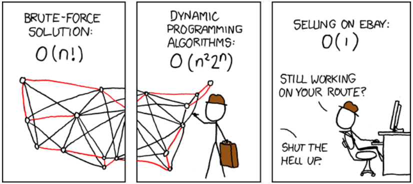
Abstract constraint programming



Abstract constraint programming



Why?

- *A framework for combining constraint solvers*
- *Constraint solving on GPUs*

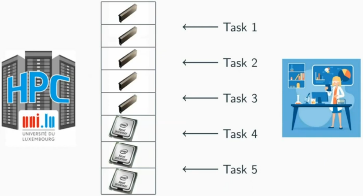
Constraint programming

- Constraint programming: we only specify what should be the solution using relations on variables (declarative programming).

Constraint programming

- Constraint programming: we only specify what should be the solution using relations on variables (declarative programming).
- But we do not program how to compute the solution.

An exemple of constraint problem



- Constraint problem: Tasks have a duration, use resources (#CPU/#GPU), and have precedence relations.
- Goal: Find a minimal schedule of the tasks on the HPC.

Scheduling problem RCPSP

NP-complete optimisation problem:

- T is a set of tasks, $d_i \in \mathbb{N}$ the duration of task i.
- P are the precedences among tasks: $i \ll j \in P$ if i must terminate before j starts.
- R is a set of resources where $k \in R$ has a capacity $c_k \in \mathbb{N}$.
- Each task i uses a quantity $r_{k,i}$ of resources k.

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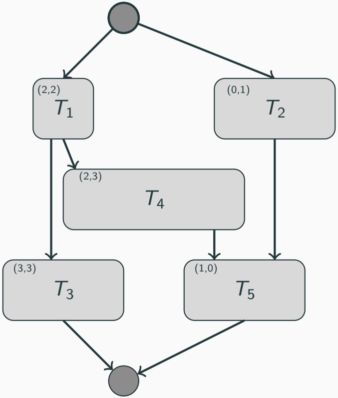
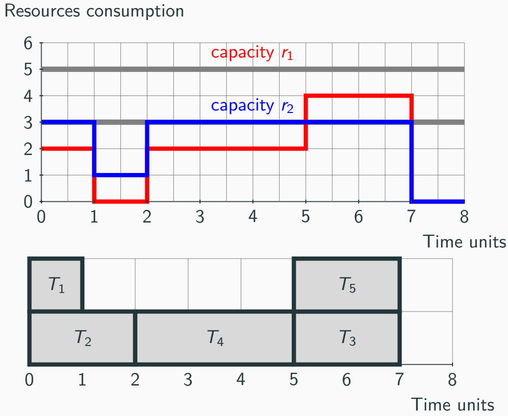
- R is a set of resources where $k \in R$ has a capacity $c_k \in \mathbb{N}$.
- Each task i uses a quantity $r_{k,i}$ of resources k.

Goal: find a (minimal) planning of tasks T that satisfies precedences in P without exceeding the capacity of available resources.

Example with 5 tasks and 2 resources





CSP & COP

Constraint satisfaction problem

A `constraint satisfaction problem` is a tuple $P = \langle X, D, C \rangle$ where:

- a *finite set of variables*, denoted by X
- $D_i \in D$ the set of values taken by each variable $x_i \in X$
- a *finite set of constraints*, denoted by C, each covering a sub-set of X such as $\forall c \in, \texttt{scope}(c) \subseteq X$.

CSP & COP

Constraint satisfaction problem

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Constraint satisfaction problem

A `constraint optimization problem` is a tuple $P = \langle X, D, C, O \rangle$ where:

- a *finite set of variables*, denoted by X
- $D_i \in D$ the set of values taken by each variable $x_i \in X$
- a *finite set of constraints*, denoted by C, each covering a sub-set of X such as $\forall c \in, \texttt{scope}(c) \subseteq X$.
- an *objective function* $\mathcal{O} = \texttt{obj}(X)$ to be maximized or minimized

Constraints model [Schutt et al.]

- Variables : $s_i \in \{0..h-1\}$ is the starting time of task i.
- Constraints :

$$\forall (i \ll j) \in P,\ s_i + d_i \leq s_j \tag{1}$$

$$\begin{aligned} &\forall j \in [1..n],\ \forall i \in [1..n] \setminus \{j\}, \\ &\qquad b_{i,j} \Leftrightarrow (s_i \leq s_j \wedge s_j < s_i + d_i) \end{aligned} \tag{2}$$

$$\forall j \in [1..n],\ r_{k,j} + \Big(\sum_{i \in [1..n] \setminus \{j\}} r_{k,i} * b_{i,j}\Big) \leq c_k \tag{3}$$

1. Temporal constraints (eq. 1)
2. Resources constraints (eq. 2 and 3): *tasks decomposition* of global constraint `cumulative`.

Abstract domain

Abstract domain

An abstract domain $\langle Abs, \leq, \sqcup, \top, \gamma, [\![.]\!], refine, split \rangle$ is a lattice such that:

- Abs is a set of elements representable in a machine.
- $\leq$ is a partial order.
- $\sqcup$ performs the *join* of two elements ("union of information").
- $\top$ is the largest element ("initial state").
- $\gamma : A \rightarrow D^\flat$ is a monotone concretization function.
- $\texttt{state} : Abs \rightarrow K$ gives the state of an element ($K = \{$ true, false, unknown $\}$).
- $[\![.]\!] : \Phi \rightarrow Abs$ is a partial interpretation function turning a constraint into an element of the abstract domain.
- $refine : Abs \rightarrow Abs$ is an extensive function, *e.g.*, $a \leq refine(a)$, refining an abstract element ("gain information").
- $split : Abs \rightarrow \mathcal{P}(Abs)$ is an extensive function dividing an abstract element into a set of sub-elements.
- $\models$: $Abs \times \Phi$: $a \models \varphi$ holds whenever $\gamma(a) \subseteq [\![\varphi]\!]^\flat$ the deduction relation, called the 'entailment'.

Interval

Interval

An interval is a pair $(l, u) \in \mathbb{Z}^2$ of the lower and upper bounds, written $[l, u]$.

Lattice of intervals

The lattice of interval $\langle \mathcal{I}, \sqsubseteq, \sqcup, \sqcap, \bot, [-\infty, \infty] \rangle$ is defined as:

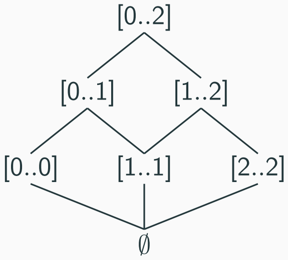
$$\mathcal{I} \triangleq \{[a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{\infty\}, a \sqsubseteq b\} \cup \{\bot\}$$

with the following operations:

- $[a, b] \sqsubseteq [c, d] \Leftrightarrow a \geq c \wedge b \leq d$.
- $[a, b] \sqcup [c, d] \triangleq [\min(a, c), \max(b, d)]$.
- $[a, b] \sqcap [c, d] \triangleq [\max(a, c), \min(b, d)]$.

Example of lattice of intervals

For the set $\{0, 1, 2\}$



Box

Box domain

- Let $\mathcal{I}$ be the lattice of integer intervals, and V a set of variables.
- Then $\text{Box} = [V \nrightarrow \mathcal{I}]$ is the abstract domain of box.

It treats constraints of the form

$$x \leq d \quad x \geq d$$

where $d \in \mathbb{Z}$ is a constant.

Octagon abstract domain

Octagonal constraint

Octagonal constraint

We call `octagagonal constraint` any constraint of the form $\pm x_i - \pm x_j \leq c$ with c is a constant from $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

We call `octagon` the set of points satisfying a conjuction of octagonal constraints.

Remark

The name `octagon` comes from the fact that, in two dimensions, our sets are `polyhedra` with at most eight sides.

Potential constraints

Potential constraint

We call `potential constraint` any constraint of the form $x_i - x_j \leq c$.

Potential constraints

Potential constraint

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Potential graphs

A conjunction of potential constraints can be represented as a directed graph $\mathcal{G}$ with nodes from $(x_0, \ldots, x_{n-1})$ and value in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

For each ordered pair of variables $x_i, x_j \in \mathcal{V}^2$, there will be an arc from x_i to x_j with weight c if the constraint $x_i - x_j \leq c$ is in constraint conjuction.

Difference Bound Matrix

Difference bound matrices

An equivalent representation for potential constraint conjuction is by means of a `Difference Bound Matrix` (DBM).

A DBM m is a $n \times n$ square matrix where n is the number of variables.

The element at line i, column j where $1 \leq i \leq n$, $1 \leq j \leq n$, denoted by m_{ij}, equals to c if there is a constraint of the form $x_i - x_j \leq c$ in our constraint conjunction and $+\infty$ otherwise.

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Remark

A DBM m can be seen as the adjacency matrix of a potential graph.

Transformation of octagonal constraints

Transformation of octagonal constraints

From the set of variables $\mathcal{V} = (x_0, \ldots, x_{n-1})$ we derive the set $\mathcal{V}' = (x'_0, \ldots, x'_{2n})$.

Each variable $x_i \in \mathcal{V}$ has both a `positive form` x'_{2i}, and a `negative form` x'_{2i+1}.

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We will encode `octagonal constraints` on $\mathcal{V}$ as `potential constraints` on $\mathcal{V}'$.

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Each variable $x_i \in \mathcal{V}$ has both a `positive form` x'_{2i}, and a `negative form` x'_{2i+1}.

We will encode `octagonal constraints` *on* $\mathcal{V}$ *as* `potential constraints` *on* $\mathcal{V}'$.

$$
\begin{aligned}
x_i - x_j \leq d &\rightsquigarrow x'_{2i} - x'_{2j} \leq d \wedge x'_{2j+1} - x'_{2i+1} \leq d \\
x_i + x_j \leq d &\rightsquigarrow x'_{2i} - x'_{2j+1} \leq d \wedge x'_{2j} - x'_{2i+1} \leq d \\
-x_i - x_j \leq d &\rightsquigarrow x'_{2i+1} - x'_{2j} \leq d \wedge x'_{2j+1} - x'_{2i} \leq d \\
x_i \leq d &\rightsquigarrow x'_{2i} - x'_{2i+1} \leq 2d \\
-x_i \leq d &\rightsquigarrow x'_{2i+1} - x'_{2i} \leq 2d
\end{aligned}
$$

Transformation of octagonal constraints

- In a `potential constraint` x'_{2i} will represent x_i while x'_{2i+1} will represent $-x_i$.
- A conjuction of `octagonal constraints` on $\mathcal{V}$ can be represented as a DBM of dimension $2 \times n$.

Example of transformation of octagonal system to potential constraint system

$$x_0 \leq 3$$
$$x_1 \leq 2$$
$$x_0 + x_1 \leq 6$$
$$-x_0 - x_1 \leq 5$$
$$-x_0 \leq 3$$

Example of transformation of octagonal system to potential constraint system

$$x_0 \leq 3$$
$$x_1 \leq 2$$
$$x_0 + x_1 \leq 6$$
$$-x_0 - x_1 \leq 5$$
$$-x_0 \leq 3$$

How to translate this octagonal system and fill the DBM ?

Example of transformation of octagonal system to potential constraint system

$$x_0 \leq 3$$
$$x_1 \leq 2$$
$$x_0 + x_1 \leq 6$$
$$-x_0 - x_1 \leq 5$$
$$-x_0 \leq 3$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	∞	∞	∞
x'_1	∞	∞	∞	∞
x'_2	∞	∞	∞	∞
x'_3	∞	∞	∞	∞

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$

$x'_0 - x'_1 \leq 6$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	∞	∞	∞
x'_1	∞	∞	∞	∞
x'_2	∞	∞	∞	∞
x'_3	∞	∞	∞	∞

We apply $x_i \leq d \rightsquigarrow x'_{2i} - x'_{2i+1} \leq 2d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$ $\qquad$ $x'_0 - x'_1 \leq 6$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	∞
x'_1	∞	∞	∞	∞
x'_2	∞	∞	∞	∞
x'_3	∞	∞	∞	∞

We apply $x_i \leq d \rightsquigarrow x'_{2i} - x'_{2i+1} \leq 2d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	∞
x'_1	∞	∞	∞	∞
x'_2	∞	∞	∞	∞
x'_3	∞	∞	∞	∞

We apply $x_i \leq d \rightsquigarrow x'_{2i} - x'_{2i+1} \leq 2d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	∞
x'_1	∞	∞	∞	∞
x'_2	∞	∞	∞	4
x'_3	∞	∞	∞	∞

We apply $x_i \leq d \rightsquigarrow x'_{2i} - x'_{2i+1} \leq 2d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$
$x_0 + x_1 \leq 6$	$x'_0 - x'_3 \leq 6$, $x'_2 - x'_1 \leq 6$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	∞
x'_1	∞	∞	∞	∞
x'_2	∞	∞	∞	4
x'_3	∞	∞	∞	∞

We apply $x_i + x_j \leq d \rightsquigarrow x'_{2i} - x'_{2j+1} \leq d \wedge x'_{2j} - x'_{2i+1} \leq d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$
$x_0 + x_1 \leq 6$	$x'_0 - x'_3 \leq 6$, $x'_2 - x'_1 \leq 6$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	∞	∞	∞	∞
x'_2	∞	6	∞	4
x'_3	∞	∞	∞	∞

We apply $x_i + x_j \leq d \rightsquigarrow x'_{2i} - x'_{2j+1} \leq d \wedge x'_{2j} - x'_{2i+1} \leq d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$
$x_0 + x_1 \leq 6$	$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$
$-x_0 - x_1 \leq 5$	$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	∞	∞	∞	∞
x'_2	∞	6	∞	4
x'_3	∞	∞	∞	∞

We apply $-x_i - x_j \leq d \rightsquigarrow x'_{2i+1} - x'_{2j} \leq d \wedge x'_{2j+1} - x'_{2i} \leq d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$
$x_0 + x_1 \leq 6$	$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$
$-x_0 - x_1 \leq 5$	$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	∞	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

We apply $-x_i - x_j \leq d \rightsquigarrow x'_{2i+1} - x'_{2j} \leq d \wedge x'_{2j+1} - x'_{2i} \leq d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$
$x_0 + x_1 \leq 6$	$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$
$-x_0 - x_1 \leq 5$	$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$
$-x_0 \leq 3$	$x'_1 - x'_0 \leq 6$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	∞	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

We apply $-x_i \leq d \rightsquigarrow x'_{2i+1} - x'_{2i} \leq 2d$

Example of transformation of octagonal system to potential constraint system

$x_0 \leq 3$	$x'_0 - x'_1 \leq 6$
$x_1 \leq 2$	$x'_2 - x'_3 \leq 4$
$x_0 + x_1 \leq 6$	$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$
$-x_0 - x_1 \leq 5$	$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$
$-x_0 \leq 3$	$x'_1 - x'_0 \leq 6$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

We apply $-x_i \leq d \rightsquigarrow x'_{2i+1} - x'_{2i} \leq 2d$

Example - Potential graph

What about the graph representation ?

$$x_0' - x_1' \leq 6$$
$$x_2' - x_3' \leq 4$$
$$x_0' - x_3' \leq 6,\ x_2' - x_1' \leq 6$$
$$x_1' - x_2' \leq 5,\ x_3' - x_0' \leq 5$$
$$x_1' - x_0' \leq 6$$

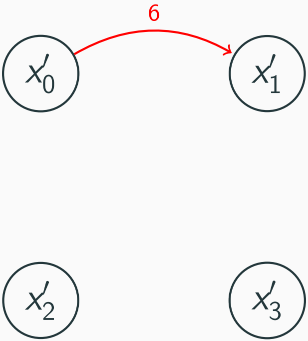
	x_0'	x_1'	x_2'	x_3'
x_0'	∞	6	∞	6
x_1'	6	∞	5	∞
x_2'	∞	6	∞	4
x_3'	5	∞	∞	∞

Example - Potential graph

What about the graph representation ?

$$x'_0 - x'_1 \leq 6$$
$$x'_2 - x'_3 \leq 4$$
$$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$$
$$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$$
$$x'_1 - x'_0 \leq 6$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

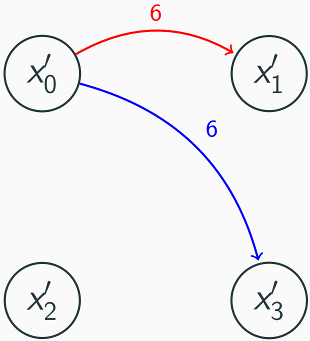


Example - Potential graph

What about the graph representation ?

$$x'_0 - x'_1 \leq 6$$
$$x'_2 - x'_3 \leq 4$$
$$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$$
$$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$$
$$x'_1 - x'_0 \leq 6$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

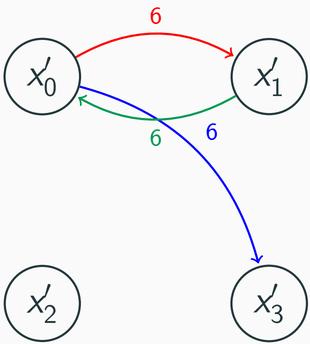


Example - Potential graph

What about the graph representation ?

$$x'_0 - x'_1 \leq 6$$
$$x'_2 - x'_3 \leq 4$$
$$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$$
$$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$$
$$x'_1 - x'_0 \leq 6$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

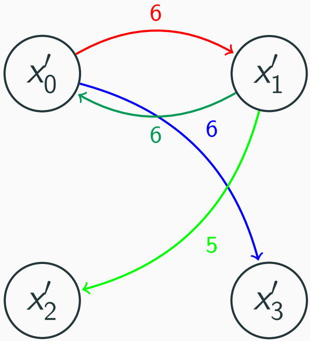


Example - Potential graph

What about the graph representation ?

$$x'_0 - x'_1 \leq 6$$
$$x'_2 - x'_3 \leq 4$$
$$x'_0 - x'_3 \leq 6, \; x'_2 - x'_1 \leq 6$$
$$x'_1 - x'_2 \leq 5, \; x'_3 - x'_0 \leq 5$$
$$x'_1 - x'_0 \leq 6$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

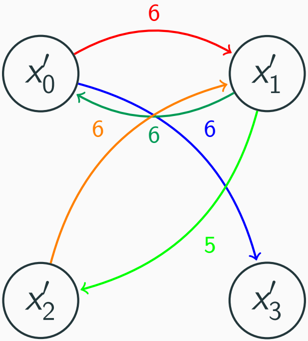


Example - Potential graph

What about the graph representation ?

$$x'_0 - x'_1 \leq 6$$
$$x'_2 - x'_3 \leq 4$$
$$x'_0 - x'_3 \leq 6,\; x'_2 - x'_1 \leq 6$$
$$x'_1 - x'_2 \leq 5,\; x'_3 - x'_0 \leq 5$$
$$x'_1 - x'_0 \leq 6$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

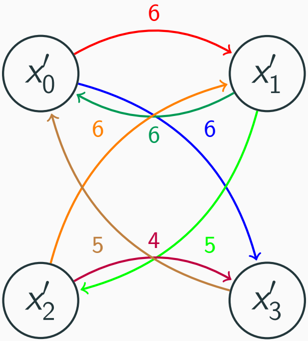


Example - Potential graph

What about the graph representation ?

$$x'_0 - x'_1 \leq 6$$
$$x'_2 - x'_3 \leq 4$$
$$x'_0 - x'_3 \leq 6,\ x'_2 - x'_1 \leq 6$$
$$x'_1 - x'_2 \leq 5,\ x'_3 - x'_0 \leq 5$$
$$x'_1 - x'_0 \leq 6$$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞



Some remarks about DBM

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	∞

- $m_{0,3} = m_{2,1} = 6.$
- $x'_0 - x'_3 \leq 6, \quad x'_2 - x'_1 \leq 6$
- $x_0 + x_1 \leq 6, \quad x_1 + x_0 \leq 6$
- DBM operations should keep entries equal.

Coherence and Consistency

Coherence

A DBM $\mathbf{m}$ is coherent iff $\forall i.j.\mathbf{m}_{i,j} = \mathbf{m}_{\bar{\jmath},\bar{\imath}}$ where $\bar{\imath} = i + 1$ if i is even and $i - 1$ otherwise.

Bar operator

The bar operation can be realised without a branch using $\bar{\imath} = i \oplus 1$.

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Bar operator

The bar operation can be realised without a branch using $\bar{\imath} = i \oplus 1$.

Consistency

A DBM **m** is consistent iff $\forall i.\mathbf{m}_{i,i} \geq 0$.

Negative cycle

Intuitively, consistency means that there is not `negative cycle` in the DBM, which corresponds to unsatisfiability.

Order

Partial order

Let m and m' two matrices of size N from two potential sets we can define the `order` operator, denoted $\leq$, as

$$m \leq m' \text{ iff } m_{i,j} \leq m'_{i,j} \quad \forall i, j \in N$$

Link with CP

The `order` allows for the removal of redundant constraints.

Join

Join $\sqcup$

Let m and m' two matrices of size N from two potential sets we can define the `join` operator as

$$m \sqcup m' = \left\{ \max\left(m_{i,j}, m'_{i,j}\right)^{i,j} \mid i, j \in N \right\}$$

Link with CP

$\sqcup$ can be seen as the `disjunction` of constraints of the form $x_0 + x_1 \leq d$.

Join - Example

Let m the matrix represented the constraint $x_0 + x_1 \leq 5$ and m' the matrix represented the constraint $x_0 + x_1 \leq 7$.

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	∞	∞	5
x'_1	∞	∞	∞	∞
x'_2	∞	5	∞	∞
x'_3	∞	∞	∞	∞

$\sqcup$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	∞	∞	7
x'_1	∞	∞	∞	∞
x'_2	∞	7	∞	∞
x'_3	∞	∞	∞	∞

$=$

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	∞	∞	7
x'_1	∞	∞	∞	∞
x'_2	∞	7	∞	∞
x'_3	∞	∞	∞	∞

Meet

Meet $\sqcap$

Let m and m' two matrices of size N from two potential sets we can define the `meet` operator as

$$m \sqcap m' = \left\{ \min \left(m_{i,j}, m'_{i,j} \right)^{i,j} \mid i, j \in N \right\}$$

Link with CP

$\sqcap$ can be seen as the `conjunction` of constraints of the form $x_0 + x_1 \leq d$.

Remark

The order $m \leq m'$ is equivalent to $m \sqcap m' = m$ and $m \leq m'$ is equivalent to $m \sqcup m' = m$

Meet - Example

Let m the matrix representing the constraint $x_0 + x_1 \leq 5$ and m' the matrix representing the constraint $x_0 + x_1 \leq 7$.

	x_0'	x_1'	x_2'	x_3'
x_0'	∞	∞	∞	5
x_1'	∞	∞	∞	∞
x_2'	∞	5	∞	∞
x_3'	∞	∞	∞	∞

$\sqcap$

	x_0'	x_1'	x_2'	x_3'
x_0'	∞	∞	∞	7
x_1'	∞	∞	∞	∞
x_2'	∞	7	∞	∞
x_3'	∞	∞	∞	∞

$=$

	x_0'	x_1'	x_2'	x_3'
x_0'	∞	∞	∞	5
x_1'	∞	∞	∞	∞
x_2'	∞	5	∞	∞
x_3'	∞	∞	∞	∞

Closure

Closure

A DBM m is `closed`, and denoted by m^* iff

- $\forall i.\mathbf{m}_{i,i} = 0$
- $\forall i, j, k.\mathbf{m}_{i,j} \leq \mathbf{m}_{i,k} + \mathbf{m}_{k,j}$

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- `Floyd-Warshall algorithm`
- Complexity of n^3 where n is the number of variables

It is basically a loop computing n matrices, m^1 to m^n, as follows

$$\begin{cases} m^0 \stackrel{\text{def}}{=} m \\ m^k_{i,j} \stackrel{\text{def}}{=} \min(m^{k-1}_{i,j}, m^{k-1}_{i,k} + m^{k-1}_{k,j}), \quad \text{if } 1 \leq i, j, k \leq n \\ m^*_{i,j} \stackrel{\text{def}}{=} \begin{cases} m^n_{i,j}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases} \end{cases}$$

Example of application of the closure operator

```
function CLOSE(m)
    for k ∈ {0, ..., 2n − 1} do
        for i ∈ {0, ..., 2n − 1} do
            for j ∈ {0, ..., 2n − 1} do
                m'_{i,j} ← min(m_{i,j}, m_{i,k} + m_{k,j})
            end for
        end for
    end for
    return m'
end function
```

Figure 1: Floyd-Warshall algorithm for computing closure of a DBM.

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end function
```

Figure 1: Floyd-Warshall algorithm for computing closure of a DBM.

	x'_0	x'_1	x'_2	x'_3
x'_0	∞	6	∞	6
x'_1	6	∞	5	∞
x'_2	∞	6	∞	4
x'_3	5	∞	∞	11

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    for k ∈ {0, ..., 2n − 1} do
        for i ∈ {0, ..., 2n − 1} do
            for j ∈ {0, ..., 2n − 1} do
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            end for
        end for
    end for
    return m'
end function
```
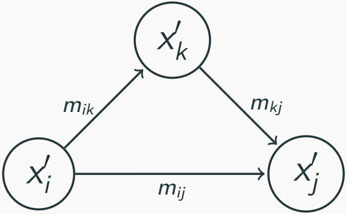

Figure 1: Floyd-Warshall algorithm for computing closure of a DBM.

	x'_0	x'_1	x'_2	x'_3
x'_0	0	6	11	6
x'_1	6	0	5	9
x'_2	9	6	0	4
x'_3	5	11	16	0

Implicit constraints

For each node x_k in turn, it checks, for all pairs (x_i, x_j), whether it would be shorter to pass through x_k instead of taking the direct arc from x_i to x_j.



This also corresponds to adding the constraints

$$x_i - x_k \leq c \wedge x_k - x_j \leq d$$

to derive the constraint (called `implicit constraint`*)*

$$x_i - x_j \leq c + d$$

The closure makes all `implicit` *constraints* `explicit`*.*

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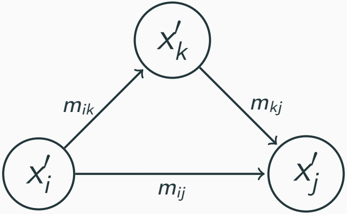

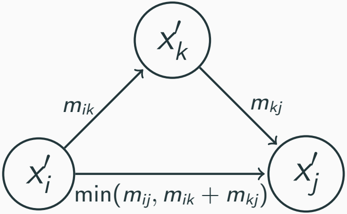

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to derive the constraint (called *implicit constraint)*

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The closure makes all *implicit* *constraints* *explicit*.

Strong closure

Closure is not sufficient for obtaining canonical form of the DBM.

- Same octagon $x_1 \leq 1 \wedge x_2 \leq 2$
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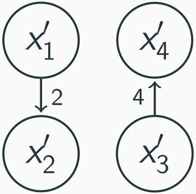

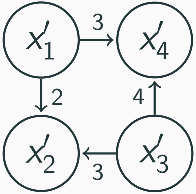

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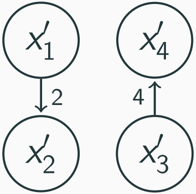

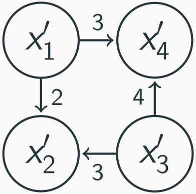

	x'_1	x'_2	x'_3	x'_4
x'_1	0	2	∞	∞
x'_2	∞	0	∞	∞
x'_3	∞	∞	0	4
x'_4	∞	∞	∞	0

	x'_1	x'_2	x'_3	x'_4
x'_1	∞	2	∞	3
x'_2	∞	∞	∞	∞
x'_3	∞	3	∞	4
x'_4	∞	∞	∞	∞

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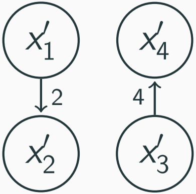

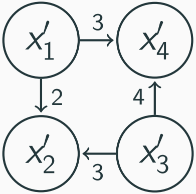

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x'_2	∞	0	∞	∞
x'_3	∞	∞	0	4
x'_4	∞	∞	∞	0

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x'_1	0	2	∞	3
x'_2	∞	0	∞	∞
x'_3	∞	3	0	4
x'_4	∞	∞	∞	0

Strong closure - Intuition

As explained before, Floyd-Warshall algorithm as performing local constraints propagations of the form

$$x'_i - x'_k \leq c \wedge x'_k - x'_j \leq d \implies x'_i - x'_j \leq c + d$$

on $\mathcal{V}'$ until no further propagation can be done.

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on $\mathcal{V}'$ until no further propagation can be done.

The idea of `strong closure` is to add another step of local constraints propagation.

$$x'_i - x'_{\bar{\imath}} \leq c \wedge x'_{\bar{\jmath}} - x'_j \leq d \implies x'_i - x'_j \leq (c + d)/2$$

such that $x'_i = -x'_{\bar{\imath}}$

so $m_{i,j}$ is replacing with $\min(m_{i,j}, (m_{i,\bar{\imath}} + m_{\bar{\jmath},j})/2)$.

Strong closure

Strong Closure

A DBM m is `strongly closed` iff

- $\mathbf{m}$ is closed
- $\forall i, j \cdot \mathbf{m}_{i,j} \leq \mathbf{m}_{i,\bar{\imath}}/2 + \mathbf{m}_{\bar{\jmath},j}/2$

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	x'_1	x'_2	x'_3	x'_4
x'_1	∞	2	∞	∞
x'_2	∞	∞	∞	∞
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	x'_1	x'_2	x'_3	x'_4
x'_1	∞	2	∞	3
x'_2	∞	∞	∞	∞
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	x'_1	x'_2	x'_3	x'_4
x'_1	0	2	∞	0
x'_2	∞	0	∞	∞
x'_3	∞	3	0	4
x'_4	∞	∞	∞	0

	x'_1	x'_2	x'_3	x'_4
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	x'_1	x'_2	x'_3	x'_4
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x'_2	∞	0	∞	∞
x'_3	∞	3	0	4
x'_4	∞	∞	∞	0

	x'_1	x'_2	x'_3	x'_4
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x'_3	∞	3	0	4
x'_4	∞	∞	∞	0

Product of abstract domain

Back to our example: three kinds of constraints in RCPSP

- octagonal constraints treated by octagon abstract domain.
- equivalence constraints treated in a specialized reduced product.
- interval constraints treated by the PP abstract domain.

$$\forall (i \ll j) \in P, s_i + d_i \leq s_j$$

$$\forall j \in [1..n], \forall i \in [1..n] \setminus \{j\}, b_{i,j} \Leftrightarrow (s_i \leq s_j \wedge s_j < s_i + d_i)$$

$$\forall j \in [1..n], r_{k,j} + \left(\sum_{i \in [1..n] \setminus \{j\}} r_{k,i} * b_{i,j} \right) \leq c_k$$

Basic product

We can define a direct product over $PP \times Oct$ as follows:

$$(p, o) \sqcup (p', o') = (p \sqcup_{PP} p', o \sqcup_{Oct} o')$$

$$[\![\varphi]\!] = \begin{cases} ([\![\varphi]\!]_{PP}, [\![\varphi]\!]_{Oct}) & \\ ([\![\varphi]\!]_{PP}, \bot_{Oct}) & \text{if } [\![\varphi]\!]_{Oct} \text{ is not defined} \\ (\bot_{PP}, [\![\varphi]\!]_{Oct}) & \text{if } [\![\varphi]\!]_{PP} \text{ is not defined} \end{cases}$$

$$refine((p, o)) = (refine(p), refine(o))$$

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$$refine((p, o)) = (refine(p), refine(o))$$

Issue: domains do not exchange information.

Reduced product via equivalence constraints [Talbot et al.]

We can improve the refinement operator of the direct product by connecting constraints from both domains via equivalence constraints.

- Let $\varphi_1 \Leftrightarrow \varphi_2$ be an equivalence constraint where $[\![\varphi_1]\!]_{PP}$ and $[\![\varphi_2]\!]_{Oct}$ are defined, then we have:

$$
\begin{aligned}
&\mathit{prop}_{\Leftrightarrow}(p, o, \varphi_1 \Leftrightarrow \varphi_2) \triangleq \\
&\begin{cases}
p \models_{PP} \varphi_1 \implies (p, o \sqcup [\![\varphi_2]\!]_{Oct}) \\
p \models_{PP} \neg\varphi_1 \implies (p, o \sqcup [\![\neg\varphi_2]\!]_{Oct}) \\
o \models_{Oct} \varphi_2 \implies (p \sqcup [\![\varphi_1]\!]_{PP}, o) \\
o \models_{Oct} \neg\varphi_2 \implies (p \sqcup [\![\neg\varphi_1]\!]_{PP}, o) \\
(p, o) \text{ otherwise}
\end{cases}
\end{aligned}
$$

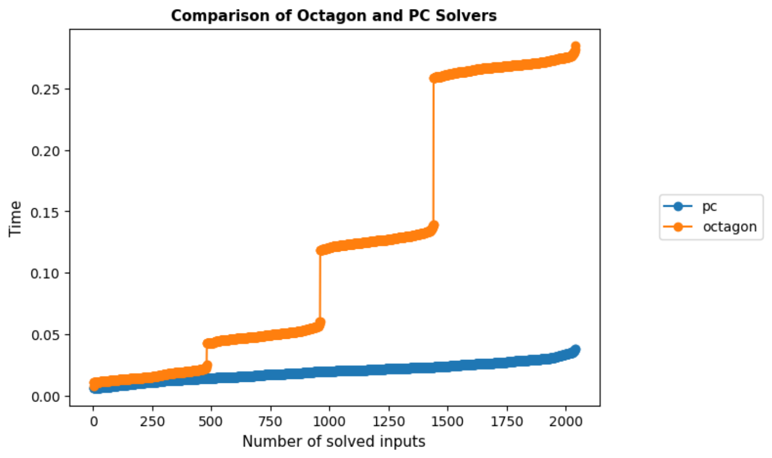
- Result: A generic reduced product to combine abstract domains with disjoint set of variables.

Experiments

Instances & Environment

- 2040 instances
  - from XCSP3 world
- STP instances (so RCPSP with only the precedence constraints)
- 120 variables
- Precision 7780 13th Gen Intel(R) Core(TM) i9-13950HX
- Timeout of 20 seconds

Octagon vs PC



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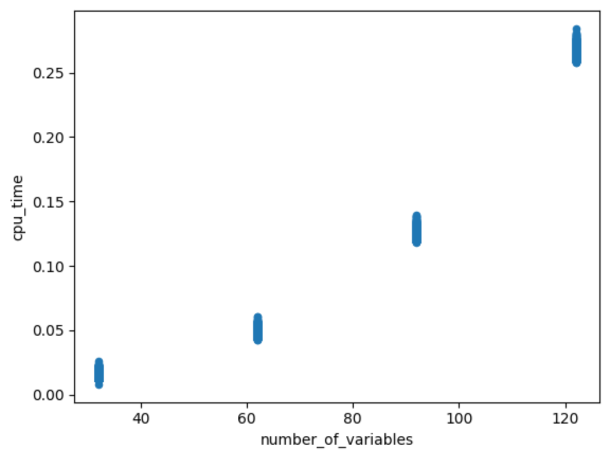

Conclusion

Conclusion

- RCPSP problem and this modelization
- Different abstract domains $\rightarrow$ Octagon abstract domain with these operators and this representation
- Based on these concepts, we model the RCPSP problem using abstract domains.
- Some experiments

Octagon Abstract Domain

Session 6

Thibault Falque

Abstract Interpretation Workshop – 20th June 2024

University of Luxembourg

UNIVERSITÉ DU LUXEMBOURG

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